

# Precise determination of resonance pole parameters through Padé approximants

Pere Masjuan,<sup>1,\*</sup> Jacobo Ruiz de Elvira,<sup>2,†</sup> and Juan José Sanz-Cillero<sup>3,‡</sup>

<sup>1</sup>*Institut für Kernphysik, Johannes Gutenberg-Universität, D-55099 Mainz, Germany*

<sup>2</sup>*Helmholtz-Institut für Strahlen- und Kernphysik, Universität Bonn, D-53115 Bonn, Germany*

<sup>3</sup>*Departamento de Física Teórica and Instituto de Física Teórica,*

*IFT-UAM/CSIC Universidad Autónoma de Madrid, Cantoblanco, Madrid, Spain*

In this work, we present a precise and model-independent method to extract resonance pole parameters from phase-shift scattering data. These parameters are defined from the associated poles in the second Riemann sheet, unfolded by the analytic continuation to the complex pole using Padé approximants. Precise theoretical parameterizations of pion-pion scattering phase shifts based on once- and twice- subtracted dispersion relations are used as input, whose functional form allows us to show the benefit and accuracy of the method. In particular, we extract from these parametrization the pole positions of the  $f_0(500)$  at  $\sqrt{s} = (453 \pm 15) - i(297 \pm 15)$  MeV, the  $\rho(770)$  at  $\sqrt{s} = (761.4 \pm 1.2) - i(71.8 \pm 1.0)$  MeV, and the pole of the  $f_2(1270)$ , located at  $\sqrt{s} = (1267.3 \pm 1.7) - i(95.0 \pm 2.3)$  MeV. The couplings of the resonances to two pions are also determined with high precision, obtaining respectively,  $3.8 \pm 0.4$  GeV,  $5.92 \pm 0.15$  and  $4.41 \pm 0.23$  GeV<sup>-1</sup>. Special attention is dedicated to the systematic treatment of the theoretical and statistical uncertainties, together with their comparison with previous determinations.

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## I. INTRODUCTION

The non-perturbative regime of Quantum Chromodynamics is characterized by the presence of hadronic resonances defined by complex  $S$ -matrix poles in unphysical Riemann sheets. Contrary to other definitions, the pole position –and the corresponding pole mass and width defined by  $s_p = (M_p - i\Gamma_p/2)^2$ – is universal and independent of the process under consideration. In addition, its residue enclose the information on the underlying process.

However, extrapolating the physical amplitude at real values of the energy, i.e., in the 1st Riemann sheet, into the complex plane and extracting resonance poles is not a trivial task. The extrapolation procedure may change drastically the value of the outcomes, specially in the case of broad states.

The simple method proposed here for the analytical continuation is given by the Padé approximants (PA) to an amplitude  $F(s)$  in terms of the total invariant squared momentum  $s$  around a point  $s_0$ , denoted by  $P_M^N(s, s_0)$  [1]:

$$P_M^N(s, s_0) = F(s) + \mathcal{O}\left((s - s_0)^{M+N+1}\right), \quad (1)$$

with  $P_M^N(s, s_0) = Q_N(s)/R_M(s)$  given by the ratio of two polynomials  $Q_N(s)$  and  $R_M(s)$  of degrees  $N$  and  $M$ , respectively [1].  $R_N(s_0)$  is chosen to be 1, without any loss of generality.

A special case of interest for the present work is given by Montessus de Ballore's theorem [2, 3]. Its simpler version states that when the amplitude  $F(s)$  is analytic inside the disk  $B_\delta(s_0)$  except for a single pole at  $s = s_p$  the sequence of one-pole PA  $P_1^N(s, s_0)$ ,

$$P_1^N(s, s_0) = \sum_{k=0}^{N-1} a_k (s - s_0)^k + \frac{a_N (s - s_0)^N}{1 - \frac{a_{N+1}}{a_N} (s - s_0)}, \quad (2)$$

converges to  $F(s)$  in any compact subset of the disk excluding the pole  $s_p$ . The constants  $a_n = \frac{1}{n!} F^{(n)}(s_0)$  are given, accordingly, by the  $n^{\text{th}}$  derivative of  $F(s)$  [1–3], being  $P_1^N(s, s_0)$  determined by the first derivatives  $F^{(0)}(s_0) = F(s_0)$ ,  $F^{(1)}(s_0) \dots F^{(N+1)}(s_0)$ .

Likewise, the PA pole and residue

$$s_p^{(N)} = s_0 + \frac{a_N}{a_{N+1}}, \quad Z^{(N)} = -\frac{(a_N)^{N+2}}{(a_{N+1})^{N+1}}, \quad (3)$$

converge to the corresponding pole and residue of  $F(s)$  for  $N \rightarrow \infty$ .

During the last years, dispersive approaches have been proved to be a very successful tool to obtain precise determinations of phase shifts and pole parameters [4–10]. However, they are based on a complicated although powerful machinery which makes them difficult to use except for a limited number of cases. In this letter, we use dispersive  $\pi\pi$  parameterizations to show how it is possible to obtain a precise and model-independent determination of resonance pole parameters using the theory of PA [1, 3], even for cases where dispersive methods cannot be easily applied.

Following the proposal in Ref. [3], Montessus' theorem is applied to the simplest case with a single-resonance pole inside the disk  $B_\delta(s_0)$ . Nonetheless, it can be generalized, ensuring the convergence of the  $P_M^N(s, s_0)$  se-

\* masjuan@kph.uni-mainz.de

† elvira@hiskp.uni-bonn.de

‡ juanj.sanz@uam.es

quence when the amplitude contains up to  $M$  poles in the disk  $B_\delta(s_0)$  [3].

Thanks to Montessus' theorem, one can use the PAs in a theoretically safe way by centering them at  $s_0 + i0^+$  over a physical brunch cut and far away enough from the branch point singularities, which will limit the theorem's applicability range in the  $s$ -variable. This allows us to unfold the 2RS, or higher sheets, through the analytical extension of  $F(s)$  from the first Riemann sheet (1RS) provided by the PA [3].

As the order of the approximant increases, the difference between consecutive orders become smaller, and the  $s_p^{(N)}$  predictions defined in Eq. (3) converge to the actual pole  $s_p$  of the amplitude  $F(s)$ . Therefore, we will consider the difference between the  $P_1^N(s, s_0)$  and  $P_1^{N-1}(s, s_0)$  as our estimator of the systematic theoretical error for  $s_p^{(N)}$  [3]:

$$\Delta s_N \equiv |s_p^{(N)} - s_p^{(N-1)}|, \quad \Delta Z_N \equiv |Z^{(N)} - Z^{(N-1)}|. \quad (4)$$

Several examples in phenomenological models together with rates of convergence for Eq. (4) can be found in Ref. [3].

## II. $\pi\pi$ -SCATTERING AND POLES

The success of our pole position determinations will rely on our capability to obtain a precise determination of the coefficients  $a_j$  appearing in Eq. (3), i.e., a sequence of  $n$ th-order derivatives with respect to  $s$  for the partial-wave at a given point.

In this work we use the recent and very precise output of the  $\pi\pi$  scattering data analysis performed in [8]. This analysis incorporates  $\pi\pi$  scattering and  $K_{l4}$  decay data –in particular, the latest results from NA48/2 [11]–, obtaining, as a first step, a simple set of unconstrained parametrization (UFD) fitted to these data for each partial wave separately up to 1.42 GeV. Consequently, this UFD parametrization is used as a starting point for a Constrained Fit to Data (CFD), in which forward dispersion relations, Roy equations, and one-subtracted coupled partial wave dispersion relations –or GKPY equations– are imposed as an additional constraint to the data fits. These relations incorporate crossing and assume analyticity in the 1RS. The interest of these CFD parametrizations is that, while describing the data, they satisfy within uncertainties dispersion relations, constraining and reducing the errors of the experimental input. This is shown in Fig. 1, where the resulting scalar-isoscalar  $\pi\pi$  phase-shift is presented. Both the UFD and CFD describe the experimental data, but in addition, the CFD satisfies the dispersive constraints imposed.

The high accuracy obtained in this dispersive analysis gives us the opportunity to use the CFD parametrization as input to obtain a precise determination of the coefficients  $a_j$  in Eq. (3), and then, to extract the pole

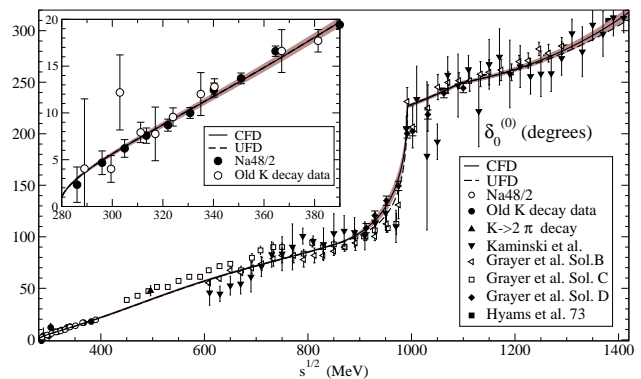


FIG. 1. S0 wave phase shift for  $\pi\pi$ -scattering experimental data together with the UFD and CFD parameterizations [8]. The dark band covers the uncertainties. In the inner top panel, we show the low-energy region and the good description of the latest NA48/2 data on  $K_{l4}$  decays, which are responsible for the small uncertainties of the UFD and CFD parameterizations.

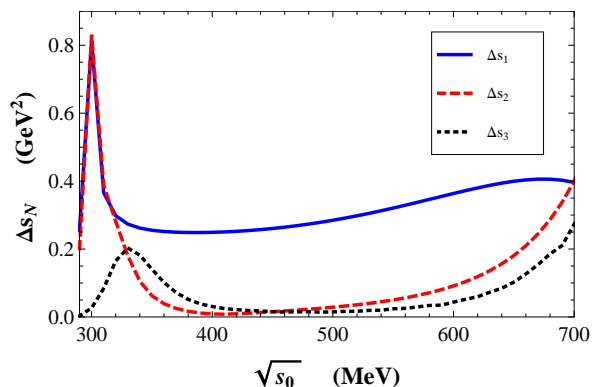


FIG. 2. Uncertainty  $\Delta s_N$  in the  $\sigma$  pole determination for the  $P_1^1(s, s_0)$  (solid blue),  $P_1^2(s, s_0)$  (dashed red) and  $P_1^3(s, s_0)$  (dotted black) approximants for different values of  $s_0$  ranging from  $2m_\pi$  up to 700 MeV. The fastest convergence is found around 500 MeV.

position of the lightest resonances appearing in  $\pi\pi$ -scattering in the  $IJ = 00, 11, 02$  channels, respectively, i.e., the  $f_0(500)$ , the  $\rho(770)$ , and the  $f_2(1270)$ . Furthermore, these parameterizations were used in [9] as input for the GKPY and Roy S0- and P-wave equation for  $\pi\pi$ -scattering, providing a model-independent continuation to the complex plane, and then, a determination of the position and residues of the second Riemann sheets poles appearing in these channels, which we can use to compare the precision of our pole extraction method, and the analysis of the errors.

Now, let us be more precise with our method and proceed to the analysis of resonances in various channels, beginning with the  $f_0(500)$  or  $\sigma$  meson.

We use the CFD  $\pi\pi$  parameterizations to obtain the value of the phase-shift  $\delta_0^0(s)$  and inelasticity  $\eta_0^0(s)$ , as well as their four first derivatives. From them, we com-

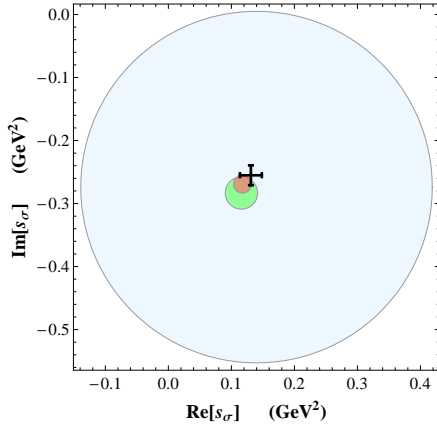


FIG. 3. Theoretical uncertainty regions  $\Delta s_N$  for the  $P_1^1(s, s_0)$  (lighter blue),  $P_1^2(s, s_0)$  (green) and  $P_1^3(s, s_0)$  (darker red) approximants. The  $1\sigma$  black error bar corresponds to the determination through the GKPY equations [9]. The PA center in this plot is  $\sqrt{s_0} = 490$  MeV.

pute the value and derivatives of the  $IJ = 00$  partial wave,

$$t_J^I(s) = (\eta_J^I(s)e^{2i\delta_J^I(s)} - 1)/(2i\rho_\pi(s)), \quad (5)$$

where  $\rho_\pi(s) = \sqrt{1 - 4m_\pi^2/s}$  is the phase-space factor,

Our method, then, proceeds as such. First, from the central values of the  $\delta_0^0(s)$   $\pi\pi$  phase-shift (Fig. 1) and its derivatives, we analyze the convergence of the theoretical uncertainty  $\Delta s_N$  of the  $P_1^N(s, s_0)$  approximants  $N = 1, 2, 3$  for different PA centers  $s_0$  between the  $\pi\pi$  and  $KK$  thresholds. Note that below the  $KK$  threshold, elastic scattering is assumed for the S0 wave in [9], so we take  $\eta_0^0(s) = 1$ .

Afterwards, the theoretical error  $\Delta s_3$  for the  $P_1^3(s, s_0)$  approximant happens to be minimized at  $\sqrt{s_0} = 490$  MeV, see Fig. 2, and the PA sequence is found to break down when  $s_0$  approaches either the  $\pi\pi$  or  $KK$  thresholds. In this way, we are able to obtain a first estimate from the  $P_1^3(s, 490^2 \text{ MeV}^2)$  without including the uncertainties of the CFD parametrization:

$$\sqrt{s_\sigma} = (453 \pm 13_{\text{sys}}) + i(297 \pm 13_{\text{sys}}) \text{ MeV}. \quad (6)$$

Even for such a broad resonance, a clear convergence can be observed in Fig. 3, where we plot the  $s_p$  theoretical uncertainty regions for the different  $P_1^N(s, s_0)$ .

Finally, in order to incorporate the statistical uncertainties coming from the input error bands, we use a MonteCarlo (MC) simulation, where for each  $s_0$  between the  $\pi\pi$  and  $KK$  thresholds,  $\{\delta^{(n)}\}$  configurations (with  $n = 0-4$ ) are generated with a distribution according to the values of the phase-shift and derivatives of [8]. However, the theoretical error in  $P_1^3(s, s_0)$  is not negligible anymore and each of these MC configurations for  $\{\delta^{(n)}\}$  do not correspond to a single point  $s_p^{(3)}$ , but to an homogeneous circle centered at that point with radius  $\Delta s_3$ , as in Fig. 3. Practically, from every point

$s_p^{(3)}$  produced, the MC generates a fixed number  $n$  of points with a uniform distribution within the circle of radius  $\Delta s_3 = |s_p^{(3)} - s_p^{(2)}|$ , centered at the given  $s_p^{(3)}$ .

The theoretical error due to the truncation of the PA sequence is of the same order as the uncertainties that stem just from the error of the phase-shift data of [8].

TABLE I. Collection of different dispersive predictions for the  $f_0(500)$  meson pole position and coupling to two pions in the 2RS.

Reference	$\sqrt{s_\sigma}(\text{MeV})$	$ g_{\sigma\pi\pi} (\text{GeV})$
[4]	$(470 \pm 30) - i(295 \pm 20)$	—
[12]	$(470 \pm 50) - i(285 \pm 25)$	—
[6]	$(457_{-13}^{+14}) - i(272_{-12.5}^{+9})$	$3.31_{-0.15}^{+0.35}$
[9]	$(457_{-13}^{+14}) - i(279_{-7}^{+11})$	$3.59_{-0.13}^{+0.11}$
[13]	$(442_{-8}^{+5}) - i(274_{-5}^{+6})$	3.37
This Work	$(453 \pm 15) - i(297 \pm 15)$	$3.8 \pm 0.4$

The combined error (theory+experiment) from the MC is minimal for  $\sqrt{s_0} = 500$  MeV and the  $P_1^3(s, s_0)$  approximant produces the  $f_0(500)$  pole position shown in Table I. In addition, we also provide in Table I the  $\sigma$  coupling to two pions defined as

$$g^2 = -16\pi Z^{(N)}(2l+1)/(2p)^{2l}, \quad (7)$$

where  $p^2 = s/4 - m_\pi^2$ , and  $Z^{(N)}$  is the pole residue given in Eq. (3), which we calculate from the  $P_1^3(s, s_0)$  in a similar way ( $|Z^{(3)}| = 0.30 \pm 0.06 \text{ GeV}^2$ ). We also show in Table I further  $f_0(500)$  determinations. In particular, the comparison with the result of [9] is specially illuminating, since it was obtained from the analytic continuation to the complex plane of the GKPY eqs, using as input, the same CFD parameterizations employed in this work. The agreement between both determinations highlight the goodness of PA as a precise method to extract resonance pole parameters.

In order to analyze the  $\rho(770)$  resonance, we repeat exactly the same procedure for the  $I = J = 1$  channel, using, this time, the CFD  $\delta_1^1(s)$  parametrization of [8]. The inelasticity is again taken as  $\eta = 1$  below the  $KK$  threshold. Without input errors, the optimal point for the PA center is  $\sqrt{s_0} = 680$  MeV, where one finds a fast convergence for the  $P_1^N(s, s_0)$  sequence:  $\Delta s_1 = 4.7 \cdot 10^4 \text{ MeV}^2$ ,  $\Delta s_2 = 1.0 \cdot 10^3 \text{ MeV}^2$ ,  $\Delta s_3 = 4.1 \text{ MeV}^2$ . Furthermore, the  $\rho$  pole position uncertainties are below 0.1 MeV for  $P_1^3(s, s_0)$  if  $\sqrt{s_0} \in [0.65 \text{ GeV}, 0.8 \text{ GeV}]$ . Incorporating the uncertainties of the parametrization through a MC, as we did before for the  $f_0(500)$ , we find that the combined error (theory+experiment) is minimized at  $\sqrt{s_0} = 740$  MeV giving the  $\rho$  pole position of Table II. Similar outcomes are obtained for the range 730–780 MeV up to 0.2 MeV variations in the error size. We also show in Table II the  $\rho$  coupling to two pions, extracted again from the pole residue ( $|Z^{(3)}| = 0.118 \pm 0.006 \text{ GeV}^2$ ). Contrary to what happened with

TABLE II. Collection of different dispersive predictions for the  $\rho$  meson resonance parameters in the 2RS.

Reference	$\sqrt{s_\rho}$ (MeV)	$ g_{\rho\pi\pi} $
[14]	$(762.5 \pm 2) - i(71 \pm 4)$	–
[4]	$(762.4 \pm 1.8) - i(72.6 \pm 1.4)$	–
[15]	$(764.1 \pm 2.7^{+4.0}_{-2.5}) - i(74.1 \pm 1.0^{+0.9}_{-3.0})$	–
[12]	$(763.0 \pm 0.2) - i(69.5 \pm 0.3)$	–
[9]	$(763.7^{+1.7}_{-1.5}) - i(73.2^{+1.0}_{-1.1})$	$6.01^{0.04}_{-0.07}$
[3]	$(763.7 \pm 1.2) - i(72.0 \pm 1.5)$	–
This Work	$(761.4 \pm 1.2) - i(71.8 \pm 1.0)$	$5.92 \pm 0.15$

the  $f_0(500)$ , in the case of the  $\rho(770)$  most of the error comes from the input uncertainties being the theoretical error essentially negligible. As in the case of the  $f_0(500)$ , the comparison with [9] shows a perfect agreement.

Finally, we want to end up with the study of the isoscalar tensor resonance  $f_2(1270)$  through the  $\pi\pi \rightarrow \pi\pi$  partial-wave scattering amplitude  $t_2^0(s)$  over the  $KK$  threshold, in the range  $\sqrt{s_0} \in [1.15 \text{ GeV}, 1.40 \text{ GeV}]$ . We safely assume this range as analytical, since previous analysis find that the  $\pi\pi$ ,  $KK$  and  $4\pi$  channels provide more than 95% of the  $f_2$  branching ratio [16]. In addition, the remaining observed decays ( $\eta\eta$ ,  $\eta\pi\pi$ ,  $K^0 K^- \pi^+ + \text{c.c.}$ ,  $\gamma\gamma$ ,  $e^+ e^-$ ) have branching ratios below 0.8% and their thresholds are not in the  $s_0$  range considered above. Channels that could introduce branch point singularities in that interval are not observed. Nonetheless, the inelasticity drops around the  $f_2(1270)$  down to  $\eta_2^0(s) \simeq 0.75$  and we cannot take  $\eta_2^0(s) = 1$  anymore in our analysis. Therefore, we use the CFD parametrization of the phase-shift  $\delta_2^0(s)$  and inelasticity  $\eta_2^0(s)$ , to compute the  $\pi\pi$  tensor isoscalar partial wave  $t_2^0(s)$  [8]. Apart of this subtlety, our analysis of this channel proceeds exactly in the same way as we did for the  $f_0(500)$  and  $\rho(770)$ . If the experimental uncertainties are dropped, the optimal PA center for the  $P_1^3(s, s_0)$  approximant is found to be  $\sqrt{s_0} = 1180 \text{ MeV}$ . Once the statistical errors are incorporated through the MC described previously (now generating also values for  $\{\eta^{(n)}\}$  in the MC), the total error turns minimal for  $\sqrt{s_0} = 1270 \text{ MeV}$ , producing the pole position in the 3rd Riemann sheet (3RS) and coupling to two pions ( $|Z^3| = 0.184 \pm 0.019 \text{ GeV}^2$ ) given in Table III, where we also add for comparison further  $f_2(1270)$  pole determinations. It is particularly interesting to compare our determination with the results of [17], where the  $f_2(1270)$  pole position was obtained from the process  $\gamma\gamma \rightarrow \pi\pi$ , and the  $\pi\pi$  CFD parameterizations of [9] was also used as input. Despite the absence of errors in [17], the comparison between both results shows again a nice agreement. The theoretical PA and the input uncertainties are found to be of the same order of magnitude.

TABLE III. Collection of different predictions for the  $f_2(1270)$  meson resonance parameters in the 3RS.

Reference	$\sqrt{s_{f_2}}$ (MeV)	$g_{f_2\pi\pi}$ ( $\text{GeV}^{-1}$ )
[18]	$(1268 \pm 6) - i(88 \pm 7)$	–
[19]	$(1283^{+6}_{-5}) - i(93^{+5}_{-1})$	–
[20]	$(1278 \pm 5) - i(102 \pm 10)$	–
[21]	$(1277 \pm 6) - i(98 \pm 8)$	–
[22]	$(1270 \pm 8) - i(97 \pm 18)$	–
[17]	$1267 - i108$	–
This Work	$(1267.3 \pm 1.7) - i(95.0 \pm 2.4)$	$4.41 \pm 0.23$

TABLE IV. Collection of predictions using the  $P_2^2(s, s_0)$  approximant.

$P_2^2(s, s_0)$	optimal $\sqrt{s_0}$ (MeV)	$\sqrt{s_P}$ (MeV)
$f_0(500)$	490	$(461 \pm 13) - i(300 \pm 11)$
$\rho(770)$	740	$(761.4 \pm 0.8) - i(71.7 \pm 0.7)$
$f_2(1270)$	1240	$(1268.0 \pm 1.7) - i(95.7 \pm 1.8)$

The next resonance in the  $IJ = 00$  channel is the  $f_0(980)$ . Its determination is more cumbersome as it is placed close to the  $KK$  threshold. Both the phase-shift and the inelasticity vary very quickly and an accurate determination of their first derivatives is rather complicated. In addition, the  $KK$  threshold puts a limit to the range of applicability of Montessus' theorem in the  $s$ -variable. This pathology can be cured by working in the  $k_K = \sqrt{s/4 - m_K^2}$  variable or with a conformal mapping  $\omega(s, s_0)$  [3, 8]. Nevertheless, in spite of being a relatively narrow resonance, we find that our  $PA$  sequences yield very unstable pole determinations: using data from different energies in the range  $\sqrt{s_0} = 900$ – $1100 \text{ MeV}$  gives place to displacements in the position of orders of magnitude and sometimes to different Riemann sheets –including 1RS–. No conclusive result was obtained from  $PA$  sequences with different numbers of poles, even reducing the range of analyzed data to  $\sqrt{s_0} = 950$ – $1010 \text{ MeV}$ .

Before concluding, we reassess the accuracy of our results by considering extensions of Montessus' theorem. Beyond the  $P_1^N(s, s_0)$  sequence, such theorem also ensures convergence for the  $P_2^N(s, s_0)$  even though only one resonance pole would lie in the convergence disk provided that the second  $PA$  pole lies outside of it, as expected. With four experimental derivatives, we can go up to the  $P_2^2(s, s_0)$ , with a theoretical error defined as the difference between the  $P_2^1(s, s_0)$  and  $P_2^2(s, s_0)$  pole determinations. The results for  $f_0(500)$ ,  $\rho(770)$ , and  $f_2(1270)$  are collected in Table IV and they are found to be in agreement with the more conservative  $P_1^N$  determinations given in Tables I–III.



### III. CONCLUSIONS

We have performed a safe and accurate determination of the lightest resonance pole parameters in the channels  $IJ = 00, 11, 02$ , respectively,  $f_0(500)$ ,  $\rho(770)$ , and  $f_2(1270)$ , by using Padé approximants to analytically extend the CFD  $\pi\pi$ -scattering parameterizations of [8] from real energies into the complex plane. With such method, we extract the pole position with a level of precision comparable to other approaches, keeping good control of both the experimental uncertainties stemming from the GKP input and the theoretical uncertainties deriving from the Padé approximant analytical extension. More precise experimental scattering data, even for a very short energy range, could easily improve our determination of the resonance parameters: in addition to a smaller statistical error, one could safely

extract a higher number of derivatives with appropriate precision and construct higher order PAs, hence decreasing the theoretical error.

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